

Tube MPC scheme based on robust control invariant set with application to Lipschitz nonlinear systems

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ABSTRACT

The paper presents a tube model predictive control (MPC) scheme of continuous-time nonlinear systems based on robust control invariant sets with respect to unknown but bounded disturbances. The cost functional of the optimization problem is not necessarily quadratic. The scheme has the same online computational burden as the standard MPC with guaranteed nominal stability. Robust stability, as well as recursive feasibility, is guaranteed if the optimization problem is feasible at the initial time instant. In particular, we consider a scheme to obtain robust control invariant sets for a class of Lipschitz nonlinear systems, and to show the effectiveness of the proposed schemes by a simple example.

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1. Introduction

Model predictive control (MPC) is an effective scheme to deal with multivariable constrained control problems. At each sampling instant, a finite horizon open-loop optimal control problem is solved which uses the current state of the plant as the initial state. A control sequence is obtained, and the first control action in this sequence is applied to the plant.

In order to achieve robustness of the obtained closed-loop systems, a controller must stabilize the considered system for all possible realizations of exogenous disturbances or model-plant mismatches. The exogenous disturbances or model-plant mismatches are uncertainties of the plant. In MPC, an intuitive approach is to solve a min–max optimization problem online in the presence of the uncertainties [1–5]. In general, such schemes are computationally intractable since the complexity of the resulting optimization problem grows exponentially with the increase of the prediction horizon [2]. Constraint tightening approaches, as introduced in [6–8], avoid computational complexity by using a nominal prediction model and tightened constraint sets. However, the constraint sets often shrink drastically because the effects of uncertainties increase exponentially with the increase of the prediction horizon. For discrete-time linear systems subject to

persistent but bounded disturbances, [9] provided a new constraint tightening scheme, namely tube MPC. Tube MPC reduces the online computational burden while having fixed tightened sets. The algorithms utilize both a state feedback control law and a control action. The control action, calculated online, steers the *nominal* system states to the equilibrium. The state feedback control law keeps the *actual* trajectory of the constrained system in disturbance invariant sets centered along the nominal trajectory. Compared with the standard MPC with guaranteed nominal stability, an *artificial* constraint is imposed and the nominal state is chosen as a new optimization variable in the tube MPC schemes. The results proposed by [9] require linearity of the considered system, and have been extended by [10,11] to some classes of discrete-time nonlinear systems, namely systems with matched nonlinearity, piecewise affine systems and systems satisfying conditions of local contraction mapping. An error system is considered in [12], which is the deviation between the actual systems and the nominal system. Based on the robust invariant set of the error system, a tube MPC of continuous-time nonlinear systems is proposed. The actual system trajectories lie in the robust invariant set along the nominal trajectory generated by a nominal MPC scheme. Tube MPC of discrete-time nonlinear systems [13] possesses two loops. A nominal MPC scheme in the inner loop generates a reference trajectory. The MPC control in the outer loop steers trajectories of the uncertain systems towards the reference trajectory. Thus, the proposed scheme can reduce the effect of external disturbances. An improved tube MPC of discrete-time linear systems is proposed in [14]. It removes the artificial

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constraint and has the same computational burden as the standard MPC with guaranteed nominal stability.

This paper presents an extension of the improved tube MPC scheme [14] of discrete time linear systems to general continuous time nonlinear systems. Further, it provides a scheme to construct a robust control invariant set for a class of Lipschitz nonlinear systems. The optimization problem considered is not limited to the quadratic cost functional. Both recursive feasibility and input-to-state stability (ISS) of the system are guaranteed if the online optimization problem has a feasible solution at the initial time instant. We highlight that one of the advantages of the proposed scheme is the reduction of the computational burden of the optimization problem. That is, it has the same online computational burden as the standard MPC with guaranteed nominal stability. Compared with the results in [12], the online optimization problem moves one of the “technical” constraints. Therefore, the proposed scheme can avoid numerical problems related to scaling issues.

The paper is structured as follows. In Section 2 we state the problem setup and preliminary results. The online optimization problem, the proposed tube MPC scheme are discussed in Section 3. Section 4 discusses the construction of a robust control invariant set for a class of Lipschitz nonlinear systems. Based on the robust control invariant set, a numerical example is given in Section 5.

1.1. Notations and basic definitions

Let \mathbb{R} denote the field of real numbers, \mathbb{R}^n the n -dimensional Euclidean space. For a vector $v \in \mathbb{R}^n$, $\|v\|$ the 2-norm. For a matrix $M \in \mathbb{R}^{n \times n}$, $\lambda_{\min}(M)$ ($\lambda_{\max}(M)$) is the smallest (largest) real part of eigenvalues of the matrix M , $\bar{\sigma}(M)$ the largest singular value of M . Moreover, $*$ is used to denote the symmetric part of a matrix, i.e., $\begin{bmatrix} a & b^T \\ b & c \end{bmatrix} = \begin{bmatrix} a & * \\ b & c \end{bmatrix}$. The operation \ominus represents the Pontryagin difference $\mathcal{A} \ominus \mathcal{B}$ of two sets $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$. $\text{Co}\{\cdot\}$ denotes the convex hull of a set, and $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors, i.e., $\langle x, y \rangle = x^T y$. The matrix I denotes the identity matrix with compatible dimension.

2. Problem setup and preliminary results

Consider a system described by a nonlinear ordinary differential equation (ODE):

$$\dot{x}(t) = f(x(t), u(t), w(t)), \quad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state of the system, $u(t) \in \mathbb{R}^{n_u}$ is the control input. The signal $w(t) \in \mathbb{R}^{n_w}$ is the exogenous disturbance or model-plant mismatch, which is unknown but bounded, and lies in a compact set,

$$\mathcal{W} := \{w \in \mathbb{R}^{n_w} \mid \|w\| \leq w_{\max}\},$$

i.e., $w(t) \in \mathcal{W}$, for all $t \geq 0$. The system is subject to constraints

$$x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U}, \quad \forall t \geq 0. \quad (2)$$

Some fundamental assumptions are stated in the following, which are similar to the general assumptions of MPC with guaranteed nominal stability [15,16], but take the disturbance input into account.

Assumption 1. $f(x, u, w) : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \rightarrow \mathbb{R}^{n_x}$ is continuously differentiable in x, u and w . Furthermore, $f(0, 0, 0) = 0$, thus $0 \in \mathbb{R}^{n_x}$ is an equilibrium of the system (1).

Assumption 2. $\mathcal{U} \subset \mathbb{R}^{n_u}$ is compact, $\mathcal{X} \subset \mathbb{R}^{n_x}$ is bounded and the point $(0, 0, 0)$ is contained in the interior of $\mathcal{X} \times \mathcal{U} \times \mathcal{W}$.

Assume that $x(t)$ can be measured in real time, and define a nominal system

$$\dot{\bar{x}}(t) := f(\bar{x}(t), \bar{u}(t), 0), \quad (3)$$

i.e., $w(t) \equiv 0, \bar{x}(t) \in \mathcal{X}$ and $\bar{u}(t) \in \mathcal{U}$.

Denote $z(t) := x(t) - \bar{x}(t)$ as the error (deviation) between the actual system (1) and the nominal system (3). The dynamics of the error system is given as

$$\dot{z}(t) = f(x(t), u(t), w(t)) - f(\bar{x}(t), \bar{u}(t), 0). \quad (4)$$

We will design a control signal which consists of a nominal input and a state feedback control, i.e.,

$$u(t) := \bar{u}(t) + \kappa(x(t), \bar{x}(t)), \quad (5)$$

with $\kappa(x(t), \bar{x}(t)) : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^{n_u}$.

Before proceeding, we introduce the definitions of the robust control invariant set, asymptotically ultimately bounded of a system.

Definition 1. A set $\Omega \subset \mathcal{X} \subset \mathbb{R}^{n_x}$ is a robust control invariant set for the error system (4) if there exists an ancillary feedback control law $\kappa(\cdot, \cdot)$ with $\kappa(\cdot, \cdot) + \bar{u}(\cdot) \subseteq \mathcal{U} \subset \mathbb{R}^{n_u}$ such that for all $z(t_0) \in \Omega$ and for all $w \in \mathcal{W}$, $z(t) \in \Omega$ for all $t \geq t_0$.

Furthermore, Ω is a *robust invariant set* of the error system (4) under the feedback control law κ .

Definition 2. A system is asymptotically ultimately bounded if a set of initial conditions of the system converges asymptotically to a bounded set.

The main objective of this paper is to find a control action $\bar{u}(t)$ and a control feedback law $\kappa(\cdot, \cdot)$ for constrained continuous-time nonlinear systems with respect to bounded disturbances, in particular for a class of Lipschitz nonlinear systems. The systems (1) under the control (5) is asymptotically ultimately bounded.

The following lemma provides us a way to construct a robust control invariant set for the error system (4).

Lemma 1 ([12]). Let $S : \mathbb{R}^{n_x} \rightarrow [0, \infty)$ be a continuously differentiable function and $\alpha_1(\|z\|) \leq S(z) \leq \alpha_2(\|z\|)$, where α_1, α_2 are class \mathcal{K}_∞ functions. Suppose $u : \mathbb{R} \rightarrow \mathbb{R}^{n_u}$ is chosen, and there exist $\lambda > 0$ and $\mu > 0$ such that

$$\frac{d}{dt}S(z) + \lambda S(z) - \mu w^T w \leq 0, \quad \forall w \in \mathcal{W}, \quad (6)$$

with $z \in \mathcal{X}$. Then, the system trajectory starting from $z(t_0) \in \Omega \subseteq \mathcal{X}$ will remain in the set Ω , where

$$\Omega := \left\{ z \in \mathbb{R}^{n_x} \mid S(z) \leq \frac{\mu w_{\max}^2}{\lambda} \right\}. \quad (7)$$

Remark 2.1. In general, $\dot{z} = f(x, u, w) - f(\bar{x}, \bar{u}, 0) \neq h(z, w)$. Therefore, it is not easy for a general nonlinear system to find a function $S(\cdot)$ and an associated control law to satisfy (6).

Assumption 3. Suppose that there exists a robust control invariant set Ω for the error system (4) with the control law $\kappa(\cdot, \cdot)$, such that Ω lies in the interior of \mathcal{X} and $\bar{u} + \kappa(x, \bar{x}) \in \mathcal{U}$ for all $x - \bar{x} \in \Omega$ and $\bar{u} \in \mathcal{U}$.

3. Tube MPC with a general cost functional

Tube MPC is proposed originally by [9] for discrete-time linear systems. It uses the repeated online solution to an optimization

problem, subject to the *nominal* dynamics (3) and the tightened constraints in which the initial state of the nominal dynamics is a decision variable. Here, we consider continuous-time nonlinear systems. For this, define the nominal cost functional

$$J(\phi(t_k), \varphi(\cdot)) := \int_{t_k}^{t_k+T_p} l(\phi(\tau), \varphi(\tau)) d\tau + E(\phi(t_k + T_p)),$$

where the stage cost $l: \mathbb{R}^{n_x \times n_u} \rightarrow \mathbb{R}$ is uniformly continuous in ϕ and φ , and satisfies $l(0, 0) = 0$. Furthermore, $l(\phi, \varphi) > 0$ for all $(\phi, \varphi) \neq (0, 0)$. The terms $\phi(\cdot)$ and $\varphi(\cdot)$ are piecewise continuous trajectories of t from time instant t_k to $t_k + T_p$. The terminal penalty function $E(\cdot)$ is positive semidefinite, and the prediction horizon $T_p \geq 0$.

Assume that κ , Ω and \mathcal{X}_f are given. For the measured actual state $x(t_k)$, the optimization problem is formulated as follows:

Problem 1.

$$\underset{\bar{u}(\cdot; \bar{x}(t_k), t_k)}{\text{minimize}} \quad J(\bar{x}(t_k), \bar{u}(\cdot; \bar{x}(t_k), t_k)) \quad (8a)$$

subject to

$$\dot{\bar{x}}(\tau; \bar{x}(t_k), t_k) = f(\bar{x}(\tau; \bar{x}(t_k), t_k), \bar{u}(\tau; \bar{x}(t_k), t_k), 0), \quad (8b)$$

$$\bar{x}(\tau; \bar{x}(t_k), t_k) \in \mathcal{X}_0, \quad \tau \in [t_k, t_k + T_p], \quad (8c)$$

$$\bar{u}(\tau; \bar{x}(t_k), t_k) \in \mathcal{U}_0, \quad \tau \in [t_k, t_k + T_p], \quad (8d)$$

$$\bar{x}(t_k + T_p; \bar{x}(t_k), t_k) \in \mathcal{X}_f, \quad (8e)$$

where $\mathcal{X}_0 := \mathcal{X} \ominus \Omega$ and $\mathcal{X}_f \subset \mathcal{X} \ominus \Omega$. Define $G := \{\kappa(x, \bar{x}) \in \mathbb{R}^{n_u} \mid x - \bar{x} \in \Omega, x \in \mathcal{X} \text{ and } \bar{x} \in \mathcal{X}_0\}$, $\mathcal{U}_0 := U \ominus G$. The set \mathcal{X}_f is the terminal set. We use $\bar{u}(\cdot; \bar{x}(t_k), t_k)$ to emphasize that the control input is determined with the state $\bar{x}(t_k)$ at time instant t_k , and $\bar{x}(\cdot; \bar{x}(t_k), t_k)$ is the trajectory of the nominal system (3) starting from the state $\bar{x}(t_k)$ at time t_k and driven by the input function $\bar{u}(\cdot; \bar{x}(t_k), t_k)$. The term $\bar{u}^*(\cdot; \bar{x}^*(t_k), t_k)$ denotes the optimal solution to the optimization problem at the time instant t_k , and the term $\bar{x}^*(\cdot; \bar{x}^*(t_k), t_k)$ is the trajectory of the nominal system. **Problem 1** is solved in discrete time with a sample time of δ , and the nominal control during the sample interval δ is

$$\bar{u}(\tau) := \bar{u}^*(\tau; \bar{x}^*(t_k), t_k), \quad \tau \in [t_k, t_k + \delta).$$

The overall applied control input for the actual system (1) during the sampling interval δ consequently is

$$u(\tau) := \bar{u}(\tau) + \kappa(x(\tau), \bar{x}^*(\tau; \bar{x}^*(t_k), t_k)), \quad \tau \in [t_k, t_k + \delta).$$

The nominal controller calculated online, generates a nominal state trajectory. The ancillary control law $\kappa(\cdot, \cdot)$ obtained offline, keeps the trajectories of the error system in the robust control invariant set Ω centered along the nominal trajectory.

The remaining task is to suitably choose the terminal set \mathcal{X}_f and the terminal penalty function $E(\cdot)$.

Definition 3 ([17,15]). Set $\mathcal{X}_f := \{\bar{x} \in \mathbb{R}^{n_x} \mid E(\bar{x}) \leq \alpha\}$ with $\alpha > 0$, and function $E(\bar{x})$ are a terminal set and a terminal penalty function, respectively, if there exists an admissible control law $\pi(\bar{x})$ such that,

B0. $\mathcal{X}_f \subseteq \mathcal{X}_0$,

B1. $\pi(\bar{x}) \in \mathcal{U}_0$, for all $\bar{x} \in \mathcal{X}_f$,

B2. $E(\bar{x})$ satisfies inequalities,

$$\alpha_3(\|\bar{x}\|) \leq E(\bar{x}) \leq \alpha_4(\|\bar{x}\|) \quad (9a)$$

$$\frac{\partial E(\bar{x})}{\partial \bar{x}} f(\bar{x}, \pi(\bar{x}), 0) + l(\bar{x}, \pi(\bar{x})) \leq 0, \quad \forall \bar{x} \in \mathcal{X}_f, \quad (9b)$$

where $\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ are class \mathcal{K}_∞ functions.

If the set \mathcal{X}_f and the function $E(\bar{x})$ satisfy B0–B2, then the set \mathcal{X}_f is a neighborhood of the origin, which is a level set of $E(\cdot)$. Furthermore, \mathcal{X}_f is invariant for the nominal system under the control $\bar{u} = \pi(\bar{x})$ since (9) holds.

Assumption 4. For the nominal system, there exist a locally asymptotically stabilizing controller $\pi(\bar{x})$, a terminal set $\mathcal{X}_f \subseteq \mathcal{X}_0$, and a continuously differentiable positive definite function $E(\bar{x})$ such that conditions B0–B2 are satisfied for all $\bar{x} \in \mathcal{X}_f$.

Associated with **Problem 1**, consider the algorithm:

Algorithm 1. Step 0. At time t_0 , set $\bar{x}(t_0) = x(t_0)$ in which $x(t_0)$ is the current state.

Step 1. At time t_k and current state $(\bar{x}(t_k), x(t_k))$, solve **Problem 1** to obtain the nominal control action $\bar{u}(t_k)$ and the actual control action $u(t_k) = \bar{u}(t_k) + \kappa(x(t_k), \bar{x}(t_k))$.

Step 2. Apply the control $u(t_k)$ to the system (1), during the sampling interval $[t_k, t_{k+1}]$, where $t_{k+1} = t_k + \delta$.

Step 3. Measure the state $x(t_{k+1})$ at the next time instant t_{k+1} of the system (1) and compute the successor state $\bar{x}(t_{k+1})$ of the nominal system (3) under the nominal control $\bar{u}(t_k)$.

Step 4. Set $(\bar{x}(t_k), x(t_k)) = (\bar{x}(t_{k+1}), x(t_{k+1}))$, $t_k = t_{k+1}$, and go to Step 1.

Notice that a similar algorithm was proposed in [14], which considers a tube MPC scheme of discrete-time linear systems. Since **Problem 1** is feasible for any $t \geq t_0$ if **Problem 1** is feasible at t_0 , see **Theorem 1** of this paper, we don't need Step 2 of the similar algorithm in [14] to check whether the optimization problem is feasible or not.

Remark 3.1. Since only the nominal model is used for the prediction and the nominal control action is calculated online in **Problem 1**, the scheme has the same online computational burden as the standard MPC with guaranteed nominal stability.

Remark 3.2. The set Ω is not only a robust invariant set of the error system (4) but also a robust invariant set of the original nonlinear system (1), which can be confirmed by choosing $\bar{x}(t) \equiv 0$ and $\bar{u}(t) \equiv 0$, for all $t \geq 0$.

The following theorem states the properties of the proposed algorithm.

Theorem 1. Suppose that κ , Ω and \mathcal{X}_f are given, and **Problem 1** is feasible at time t_0 . Then,

- (1) **Problem 1** is feasible for all $t > t_0$,
- (2) according to **Algorithm 1**, the trajectory of the system (1) under MPC control law is asymptotically ultimately bounded,
- (3) the closed-loop system is input-to-state stable.

Proof. (1) Since only the “measured” state of the nominal system and the nominal system dynamics are used to solve **Problem 1** at the next time instant, the online optimization does not depend on the disturbances at all. Thus, recursive feasibility is guaranteed, provided that **Problem 1** has a feasible solution at the initial time instant [15].

(2) and (3) Because of the asymptotic stability of the nominal system [15], there exists a class \mathcal{KL} function $\beta(\|\bar{x}\|, t)$ [18] such that

$$\|\bar{x}(t)\| \leq \beta(\|\bar{x}(t_0)\|, t), \quad \forall t \geq t_0.$$

Due to $S(z(t)) \leq \frac{\mu w_{\max}^2}{\lambda}$ for all $t \geq t_0$ and $z(t) \in \Omega$, there exists a class \mathcal{K} function such that

$$\|z(t)\| \leq \gamma \left(\sup_{t_0 \leq \tau \leq t} \|w(\tau)\| \right), \quad \forall t \geq t_0.$$

Since $x(t) = \bar{x}(t) + z(t)$ and $\bar{x}(t_0) = x(t_0)$,

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t) + \gamma \left(\sup_{t_0 \leq \tau \leq t} \|w(\tau)\| \right), \quad \forall t \geq t_0.$$

Therefore, the solution of the system (1) under the MPC control according to Algorithm 1 is asymptotically ultimately bounded and the closed-loop system is input-to-state stable [18]. \square

Algorithm 1 can be implemented in a parallel/offline way if the initial state $x(t_0)$ is known *a priori*. That is, calculate $\bar{u}(t_i)$, $i \in [0, \infty)$, and store it for future use.

Algorithm 2 (Parallel/Offline).

- Step 0. At time t_0 , set $\bar{x}(t_0) = x(t_0)$ in which $x(t_0)$ is the current state.
 Step 1. At time t_k , solve Problem 1 to obtain the nominal control action $\bar{u}(t_k)$, and store it.
 Step 2. Compute the state $\bar{x}(t_{k+1})$ of the nominal system under the nominal control $\bar{u}(t_k)$, where $t_{k+1} := t_k + \delta$.
 Step 3. Set $\bar{x}(t_k) := \bar{x}(t_{k+1})$, $t_k := t_{k+1}$, and go to Step 1.

(Online)

- Step a. Apply the control $u(t_k) := \bar{u}(t_k) + \kappa(x(t_k), \bar{x}(t_k))$ to the system being controlled, during the sampling interval $[t_k, t_{k+1})$, where $t_{k+1} := t_k + \delta$.
 Step b. Measure the state $x(t_{k+1})$ at the next time instant t_{k+1} of the system being controlled.
 Step c. Set $x(t_k) := x(t_{k+1})$, $t_k := t_{k+1}$, and go to Step a.

As it has been shown, the robust control invariant set plays an important role in the tube MPC scheme. In the next section, we provide sufficient conditions for the calculation of a quadratic Lyapunov function $S(z) = z^T P z$ and an ancillary linear feedback control law Kz for a class of Lipschitz nonlinear systems based on Lemma 1. Note that both the robust control invariant set and the ancillary feedback control law are calculated *offline*.

4. Robust control invariant set for a class of Lipschitz nonlinear systems

Consider the following continuous-time nonlinear system

$$\dot{x}(t) = Ax(t) + g(x(t)) + Bu(t) + B_w w(t), \quad (10)$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, $w(t) \in \mathbb{R}^{n_w}$, and $g(x) : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$ represents a nonlinear function that is continuously differentiable in x .

The nonlinear function $g(x)$ is called a Lipschitz function in the set \mathcal{X} with respect to x if there exists a constant $L > 0$ such that

$$\|g(x_1) - g(x_2)\| \leq L\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathcal{X}, \quad (11)$$

where the smallest constant L satisfying (11) is known as the Lipschitz constant. The associated nominal system is

$$\dot{\bar{x}}(t) = A\bar{x}(t) + g(\bar{x}(t)) + B\bar{u}(t). \quad (12)$$

Chosen $u(t) := \bar{u}(t) + K(x(t) - \bar{x}(t))$, $K \in \mathbb{R}^{n_u \times n_x}$, the dynamics of the error system are

$$\dot{z}(t) = (A + BK)z(t) + B_w w(t) + [g(x(t)) - g(\bar{x}(t))]. \quad (13)$$

Lemma 2. Suppose that there exist positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, non-square matrix $Y \in \mathbb{R}^{n_u \times n_x}$, and scalars $\lambda_0 > \lambda > 0$ and $\mu > 0$ such that

$$\begin{bmatrix} (AX + BY)^T + AX + BY + \lambda_0 X & B_w \\ * & -\mu I \end{bmatrix} \leq 0 \quad (14)$$

and

$$L \leq \frac{(\lambda_0 - \lambda)\alpha_{\min}(P)}{2\|P\|}. \quad (15)$$

Then, the set $\Omega := \{x \in \mathbb{R}^{n_x} \mid x^T P x \leq \frac{\mu w_{\max}^2}{\lambda}\}$ is a robust invariant set for the error system (13), where $u(t) - \bar{u}(t) := Kz(t)$, $S(z) := z^T P z$, $P := X^{-1}$ and $K := YX^{-1}$.

Proof. First, consider the system

$$\dot{s}(t) = (A + BK)s(t) + B_w w(t).$$

Define $\tilde{S}(s(t)) := s(t)^T P s(t)$, and denote $H(s(t)) := \dot{\tilde{S}}(t) + \lambda_0 \tilde{S}(t) - \mu w(t)^T w(t)$. Then,

$$\begin{aligned} H(s(t)) &= s(t)^T [(A + BK)^T P + P(A + BK)]s(t) \\ &\quad + w(t)^T B_w^T P s(t) + s(t)^T P B_w w(t) \\ &\quad + \lambda_0 s(t)^T P s(t) - \mu w(t)^T w(t). \end{aligned}$$

Multiplying (14) from left and right sides with $\text{diag}\{P, I\}$ and substituting $P = X^{-1}$, $K = YX^{-1}$, we obtain that

$$\begin{bmatrix} (A + BK)^T P + P(A + BK) + \lambda_0 P & P B_w \\ B_w^T P & -\mu I \end{bmatrix} \leq 0. \quad (16)$$

The inequality (16) is sufficient to $H(s(t)) \leq 0$, which can be confirmed by multiplying (16) from both sides with $\begin{bmatrix} s(t) \\ w(t) \end{bmatrix}$ and $\begin{bmatrix} s^T(t) & w^T(t) \end{bmatrix}^T$, respectively. Because of Lemma 1, there exists a set $\Omega_0 := \{s \in \mathbb{R}^{n_x} \mid s^T P s \leq \frac{\mu w_{\max}^2}{\lambda_0}\}$ such that Ω_0 is a robust invariant set for the system $\dot{s}(t) = (A + BK)s(t) + B_w w(t)$.

Denote $M(z(t)) = \dot{S}(t) + \lambda S(t) - \mu w(t)^T w(t)$, $\lambda < \lambda_0$. For the error system (13),

$$\begin{aligned} M(z(t)) &= \dot{S}(t) + \lambda S(t) - \mu w(t)^T w(t) \\ &= z(t)^T [(A + BK)^T P + P(A + BK)]z(t) \\ &\quad + 2w(t)^T B_w^T P z(t) + \lambda z(t)^T P z(t) - \mu w(t)^T w(t) \\ &\quad + 2[g(x(t)) - g(\bar{x}(t))]^T P z(t), \\ &= H(z(t)) + (\lambda - \lambda_0)z(t)^T P z(t) \\ &\quad + 2[g(x(t)) - g(\bar{x}(t))]^T P z(t). \end{aligned}$$

Since $H(z(t)) \leq 0$ and $\alpha_{\min}\|z(t)\|^2 \leq z(t)^T P z(t) \leq \alpha_{\max}\|z(t)\|^2$,

$$\begin{aligned} M(z(t)) &\leq (\lambda - \lambda_0)z(t)^T P z(t) + 2[g(x(t)) - g(\bar{x}(t))]^T P z(t) \\ &\leq (\lambda - \lambda_0)\alpha_{\min}(P)\|z(t)\|^2 + 2[g(x(t)) - g(\bar{x}(t))]^T P z(t). \end{aligned}$$

Due to (11) and (15), we have

$$\begin{aligned} M(z(t)) &\leq (\lambda - \lambda_0)\alpha_{\min}(P)\|z(t)\|^2 + 2L\|P\|\|z(t)\|^2 \\ &= (2L\|P\| + (\lambda - \lambda_0)\alpha_{\min}(P))\|z(t)\|^2 \leq 0. \end{aligned}$$

Because of Lemma 1, this is sufficient that the set Ω is a robust invariant set for the error system (13). \square

Remark 4.1. Tube MPC of the discrete-time nonlinear system $x_{k+1} = g(x_k) + Bu_k + w_k$ is discussed in [11], where w_k is bounded and the system satisfies the conditions of local contraction mapping, see Assumption 3 of [11].

Remark 4.2. Consider linear systems with norm-bounded uncertainty,

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + B_w w(t). \quad (17)$$

Suppose that $\bar{\sigma}(\Delta A) \leq L$, where

$$\bar{\sigma}(\Delta A) := \sup_{x(t_1), x(t_2)} \frac{\|\Delta A x(t_1) - \Delta A x(t_2)\|}{\|x(t_1) - x(t_2)\|}, \quad (18)$$

for all $x(t_1) \neq x(t_2)$, is the largest singular value of ΔA [19]. Compared (18) with (11), we know that Lemma 2 also holds for the system (17).

The admissible Lipschitz constant L is always small since $\alpha_{\min}(P) \leq \|P\|$, see (15). In order to reduce the conservativeness, we can resort to the concept of one-sided Lipschitz continuity.

Definition 4. A nonlinear function $\phi(x) : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^n$ is said to be one-sided Lipschitz continuous in a set \mathcal{D} if there exists a $\rho \in \mathbb{R}$ such that for all $x_1, x_2 \in \mathcal{D}$,

$$\langle \phi(x_1) - \phi(x_2), x_1 - x_2 \rangle \leq \rho \|x_1 - x_2\|^2,$$

where ρ is called a one-sided Lipschitz constant.

Any Lipschitz function is a one-sided Lipschitz function, since

$$|\langle \phi(x_1) - \phi(x_2), x_1 - x_2 \rangle| \leq \|\phi(x_1) - \phi(x_2)\| \|x_1 - x_2\| \leq |\rho| \|x_1 - x_2\|^2.$$

However, the converse is not true in general.

Remark 4.3 ([20]). The one-sided Lipschitz constant has some properties:

- (i) It can be positive, zero or even negative.
- (ii) It is *smaller* than, or equal to, the Lipschitz constant.

Corollary 1. Suppose that there exist a positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, a non-square matrix $Y \in \mathbb{R}^{n_u \times n_x}$, and scalars $\lambda_0 > \lambda > 0$ and $\mu > 0$ such that

$$\begin{bmatrix} (AX + BY)^T + AX + BY + \lambda_0 X & B_w \\ * & -\mu I \end{bmatrix} \leq 0,$$

and $Pf(x(t))$ is one-sided Lipschitz continuous, i.e., $\langle Pg(x(t)) - Pg(\bar{x}(t)), x(t) - \bar{x}(t) \rangle \leq \rho \|x(t) - \bar{x}(t)\|^2$ with $P := X^{-1}$.

If $\rho \leq \frac{(\lambda_0 - \lambda)\alpha_{\min}(P)}{2}$, where $\alpha_{\min}(P)$ is the smallest eigenvalue of the positive definite matrix P . Then, the set $\Omega := \{z \in \mathbb{R}^{n_x} \mid z^T P z \leq \frac{\mu w_{\max}^2}{\lambda}\}$ is a robust invariant set for the error system (13), where $u(t) - \bar{u}(t) := Kz(t)$, $S(z) := z^T P z$ and $K := YX^{-1}$.

Sketch of the proof.

$$\begin{aligned} M(z(t)) &\leq (\lambda - \lambda_0)z(t)^T P z(t) + 2[g(x(t)) - g(\bar{x}(t))]^T P z(t) \\ &\leq (\lambda - \lambda_0)\alpha_{\min}(P)\|z(t)\|^2 + 2\rho\|z(t)\|^2 \\ &= (2\rho + (\lambda - \lambda_0)\alpha_{\min}(P))\|z(t)\|^2. \end{aligned}$$

Since $\rho \leq \frac{(\lambda_0 - \lambda)\alpha_{\min}(P)}{2}$, $M(z(t)) \leq 0$. Because of Lemma 1, the set Ω is a robust invariant set for the error system (13). \square

In the next section, we exemplify the derived results considering a numerical example.

5. Illustrative example

Consider the system described by

$$\dot{x}(t) = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} x(t) + g(x(t)) + \begin{bmatrix} 0.5 \\ -2 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t), \quad (19)$$

with $g(x) = [0 \quad -0.25x_2^3]^T$. Origin of this system is open-loop unstable and its linearized system is stabilizable. Assume that x_1 and x_2 can be measured instantaneously, and the control constraint is

$$-2 \leq u(t) \leq 2, \quad \forall t \geq 0.$$

The disturbance is bounded by $w(t) \in \mathbb{W} := \{w \in \mathbb{R} \mid \|w\| \leq 0.1\}$. Choose the stage penalty function as $l(x, u) = x^T Q x + u^T R u$, where the penalty matrices $Q = \text{diag}(0.5, 0.5)$ and $R = 1$.

According to the Mean-value theorem, $g(x)$ is a region Lipschitz function with a Lipschitz constant $L = 0.75x_{2,\max}^2$ provided that $x_2 \in [-x_{2,\max}, x_{2,\max}]$. Since the admissible Lipschitz constant is very small if Lemma 2 is adopted to obtain a robust control invariant set, we resort to the one-sided Lipschitz constant. The following remark will be used in the example.

Remark 5.1 ([21]). If a scalar function $h(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable with respect to x , then, for any $x, \bar{x} \in \mathbb{R}^n$ there exists $\xi \in \text{Co}(x, \bar{x})$ such that

$$h(x) - h(\bar{x}) = \left(\frac{\partial h}{\partial x_1}(\xi), \frac{\partial h}{\partial x_2}(\xi), \dots, \frac{\partial h}{\partial x_n}(\xi) \right) (x - \bar{x}).$$

According to the remark, for any $P_0 = \text{diag}(\alpha_1, \alpha_2)$ with $\alpha_1 > 0$ and $\alpha_2 > 0$, there exists a non-zero $\xi \in (\min(x_2, \bar{x}_2), \max(x_2, \bar{x}_2))$ such that

$$\begin{aligned} \langle P_0(g(x) - g(\bar{x})), x - \bar{x} \rangle &= \alpha_2(-0.25x_2^3 + 0.25\bar{x}_2^3)(x_2 - \bar{x}_2) \\ &= -0.75\alpha_2\xi^2(x_2 - \bar{x}_2)^2 \leq 0. \end{aligned}$$

That is, $P_0 g(x)$ is a one-sided Lipschitz nonlinearity with the one-sided Lipschitz constant $\rho = 0$. In this case, the robust control invariant set for the linear system $\dot{x}(t) = (A + BK)x(t) + B_w w(t)$ is also a robust control invariant set for the system (13). The ancillary control law $Kx = [-1.3693 \quad 5.1273]x$ guarantees that the set Ω is robustly invariant for the error system (13), where $\Omega = \{x \in \mathbb{R}^2 \mid x^T P x \leq 1\}$ with $P = \text{diag}(39.0251, 486.0402)$. Both the terminal control law and the terminal penalty matrix are yielded by the solution of a convex optimization problem, see [22]. Further, $\pi(x) = [-1.1456 \quad 1.3925]x$ and $E(x) = x^T \begin{bmatrix} 7.9997 & -12.2019 \\ -12.2019 & 27.0777 \end{bmatrix} x$. The terminal set of the optimization problem is $\mathcal{X}_f = \{x \in \mathbb{R}^2 \mid E(x) \leq 10\}$. The open-loop optimization problem described by Problem 1 is solved in discrete time with a sample time of $\delta = 0.1$ time units and a prediction horizon of $T_p = 1.5$ time units. Here, only Algorithm 1 is adopted. Fig. 1 shows the state trajectory of the considered system starting from state $[0.6 \quad -0.6]^T$ with the disturbances $w(t) \equiv 0.1$, for all $t \geq 0$. The dashed line shows the trajectory of the nominal system, and the solid line shows the trajectory of the actual system. As it can be seen, the trajectory of the actual system under persistent but bounded disturbances, remains in the “robust control invariant sets” centered along the nominal trajectory. Furthermore, the system state remains in the robust control invariant set around the origin while the time approaches infinity. Fig. 2 shows the state trajectory of the considered system starting from the state $[3.5 \quad -2.5]^T$ with the disturbance $w(t) \equiv 0.1$, for all $t \geq 0$. Since the trajectory of the actual system is close to the trajectory of the nominal system, it is hard to distinguish them clearly. From both Figs. 1 and 2, the control input injected to the actual system approaches, but is actually not equal to, zero with time increasing.

To deal with the control problem, we can also resort to MPC with restricted constraints originally proposed in [6,23]. The corresponding optimization problem for continuous time system is

Problem 2.

$$\begin{aligned} &\text{minimize} \quad J(\bar{x}(t_k), \bar{u}(\cdot; \bar{x}(t_k), t_k)) \\ &\bar{u}(\cdot; \bar{x}(t_k), t_k) \end{aligned} \quad (20a)$$

subject to

$$\dot{\bar{x}}(\tau; \bar{x}(t_k), t_k) = f(\bar{x}(\tau; \bar{x}(t_k), t_k), \bar{u}(\tau; \bar{x}(t_k), t_k), 0), \quad (20b)$$

$$\bar{x}(\tau; \bar{x}(t_k), t_k) \in \mathcal{X} \ominus \mathcal{H}_\tau, \quad \tau \in [t_k, t_k + T_p], \quad (20c)$$

$$\bar{u}(\tau; \bar{x}(t_k), t_k) \in \mathcal{U}, \quad \tau \in [t_k, t_k + T_p], \quad (20d)$$

$$\bar{x}(t_k + T_p; \bar{x}(t_k), t_k) \in \mathcal{X}_f \ominus \mathcal{H}_{T_p}, \quad (20e)$$

$$\bar{x}(t_0) = x_0, \quad (20f)$$

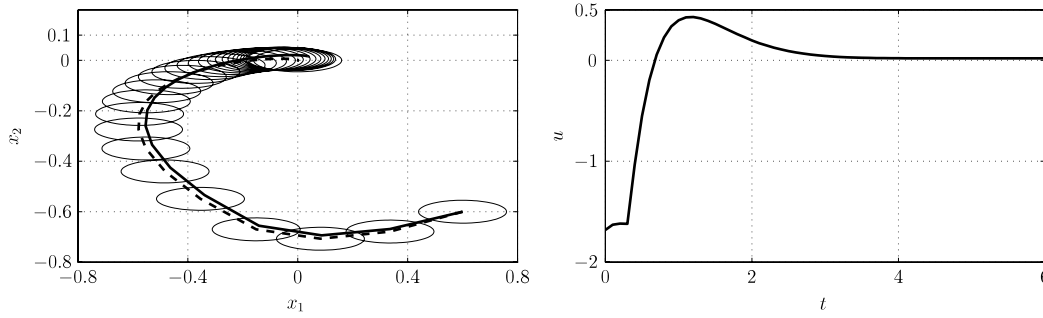


Fig. 1. Time profiles for the closed-loop system from $[0.6 \ -0.6]^T$, solid line: trajectory of real system, dashed line: trajectory of nominal system. Left: trajectory of the system under the MPC control law, Right: control input.

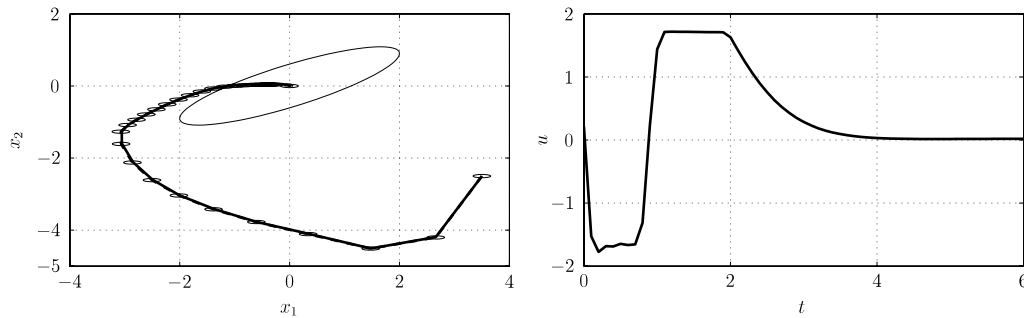


Fig. 2. Time profiles for the closed-loop system from $[3.5 \ -2.5]^T$, solid line: trajectory of real system, dashed line: trajectory of nominal system. Left: trajectory of the system under the MPC control law, Right: control input.

where the set \mathcal{H}_t is as follows [18]

$$\mathcal{H}_t := \left\{ x \in \mathbb{R}^{n_x} \mid x^T x \leq \frac{\|w\|}{L_0} (e^{L_0 t} - 1) \right\}$$

and L_0 is the Lipschitz constant of the system (1).

For the system (19), in the set $X_h := \{x \in \mathbb{R}^2 \mid -3.5 \leq x_1, x_2 \leq 3.5\}$, $L_0 = 6.1302$. If the prediction horizon $T_p \geq 0.5$, the set $X_f \ominus \mathcal{H}_{T_p}$ is empty. Thus, the largest prediction horizon of Problem 2 is 0.4. Together with the volume of the terminal set X_f being larger than $X_f \ominus \mathcal{H}_{T_p}$, Problem 1 has a larger feasible set. Furthermore, for the initial state $[3.5 \ -2.5]$, Problem 2 has no feasible solution at all even if we choose $T_p = 0.4$.

6. Conclusions

We have derived a tube MPC scheme for continuous-time nonlinear systems subject to bounded disturbances based on the robust control invariant sets. An error system is defined, which is the deviation of the actual system from the nominal system. An ancillary control law for a class of Lipschitz nonlinear systems is determined off-line which keeps a set being a robust invariant set for the error systems. The corresponding optimization problem has the same computational burden as the standard MPC with guaranteed nominal stability. The optimization problem is solved online, and its solution defines the nominal trajectory. The actual trajectory of the system under the proposed tube MPC control law is in the sets centered along the nominal trajectory. Furthermore, it had been shown that both feasibility and input-to-state stability of the closed-loop system are guaranteed if the considered optimization problem is initially feasible. The proposed scheme is illustrated by a simple example.

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